Available online at www.sciencedirect.com
ScienceDirect
Procedia Economics and Finance 14 (2014) 273 – 279
Procedia Economics and Finance
www.elsevier.com/locate/procedia

International Conference On Applied Economics (ICOAE) 2014

Note on the measures of dependence in terms of copulas

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Abstract

The dependence structure among each risk factors has been an important topic for researches both from theoretical and applied standpoints. To measure such dependence, several characteristic quantities have been already introduced and widely employed, which include, for instance, the population version of Kendall's tau (τ) and/or Spearman's rho (ρ). Copulas, on the other hand, are well known tools for understanding the dependence relation among random variables, and the above τ and ρ are expressed in terms of copulas. In this note, we generalize these expressions. We also compute the extended formula for the Archimedean copulas as well as its generalized copulas, and pursue the possibility of its applications.

Selection and/or peer-review under responsibility of the Organizing Committee of ICOAE 2014

Keywords: Copulas; Archimedean copulas; measure of dependence.

1. Introduction

The analysis of the dependence relations between random events has been one of the most important subjects for researches in probability and statistics both from theoretical and applied viewpoints. In order to quantitatively measure these relations, several characteristic notions have been already introduced and widely employed so far. To name a few, the population version of Kendall's tau which is denoted by τ and/or Spearman's rho which is denoted by ρ are both well known measures of dependence.

A copula function, on the other hand, is introduced as a tool for understanding a possibly nonlinear dependence structure among random variables. Copulas make a link between multivariate joint distributions and univariate marginal distributions, and due to their flexibilities, copulas have been applied in many situations from the financial risk management to the seismology and so on. The study of copulas thus has been very popular these days. We refer for instance to [2][3][4][5][8][12] and the references cited therin. We also mention an excellent monograph of R.B. Nelsen [9]. In addition, a well known book by A.J. McNeil, R. Frey, and P. Embrechts [7] contains the part of the theory of copulas.

In this note, we extend and generalize the formula of dependence relations involving copulas which includes the Kendall's tau (τ) and Spearman's rho (ρ). We compute these new formula for the Archimedean copulas as well as its

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2212-5671
Selection and/or peer-review under responsibility of the Organizing Committee of ICOAE 2014
doi:10.1016/S2212-5671(14)00712-6

generalized version, the so-called the generalized Archimedean (GA) copulas. The latter family is recalled in the next section. Both families contain generators and our contribution is that the formula is provided with the use of these generators, which makes the computation much simpler. We exhibit several examples of calculation employing this newly obtained formula.

2. Copulas

2.1. Definition and basic properties of copulas

We first begin with recalling the definition of copulas in the case of bivariate joint distribution.

Definition. A function C defined on $I^2 := [0,1]\times[0,1]$ and valued in $I := [0,1]$ is said to be a copula if the following conditions are satisfied.

(i) For every $(u,v)\in I^2$,

$$C(u,0)=C(0,v)=0,\quad C(u,1)=u\quad \text{and}\quad C(1,v)=v. \tag{1}$$

(ii) For every $(u_i,v_i)\in I^2$ $(i=1,2)$ with $u_1\le u_2$ and $v_1\le v_2$,

$$C(u_1,v_1)-C(u_1,v_2)-C(u_2,v_1)+C(u_2,v_2)\ge 0. \tag{2}$$

The requirement (2) is referred to as *the* 2*-increasing condition*. It is noted that a copula is a continuous function by its definition.

The well-know result due to A. Sklar [11], who employed the term “copula” almost for the first time, gives the fundamental and important property of copulas. We here recall Sklar’s theorem in bivariate case, for completeness of our presentation.

Theorem (Sklar’s theorem). *Let H be a bivariate joint distribution function with marginal distribution functions F and G; namely,*

$$\lim_{x\to\infty}H(x,y)=G(y),\qquad \lim_{y\to\infty}H(x,y)=F(x).$$

Then there exists a copula, which is uniquely determined on $\mathrm{Ran}F\times \mathrm{Ran}G$, such that

$$H(x,y)=C(F(x),G(y)). \tag{3}$$

Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (3) is a bivariate joint distribution function with margins F and G.

Next basic estimates are called the Fréchet-Hoeffding bounds.

Theorem (Fréchet-Hoeffding bounds). *For every copula C and every $(u,v)\in I^2$, we have*

$$\max\{u+v-1,0\}\le C(u,v)\le \min\{u,v\}. \tag{4}$$

For the proof of above theorems, we refer for instance to a book by Nelsen [9].

2.2. Archimedean copulas

An important class of copulas is provided by the so-called Archimedean copulas. We recall its definition for the readers’ sake.

Let φ be a convex function defined on I and valued in $[0,\infty]$ such that φ is strictly decreasing and fulfills $\varphi(1)=0$. Let $\varphi^{[-1]}$ denote the pseudo-inverse of φ; that is, $\mathrm{Dom}\varphi^{[-1]}=[0,\infty]$, $\mathrm{Ran}\varphi^{[-1]}=I$, and

$$\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1}(t) & (0\le t\le \varphi(0))\\ 0 & (\varphi(0)\le t\le\infty).\end{cases}$$

It is then proved that the function C defined on I^2 by

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \tag{5}$$

satisfies the properties (1)(2) in Definition above, and thus C gives a copula.

Copulas of the form (5) are called Archimedean copulas and the function φ is called a generator of the copula.

The class of Archimedean copula admits a wide range of applications. We here present several exmples of Archimedean copulas with their generators.

Example 1. Let $\varphi(t) = \theta^{-1}(t^{-\theta} - 1)$ with $\theta \in [-1,\infty) \setminus \{0\}$. We then have

$$C(u,v) = (\max\{u^{-\theta} + v^{-\theta} - 1, 0\})^{-1/\theta}.$$

This is a Clayton family of copulas.

Example 2. Let $\varphi(t) = (\log t)^{\theta}$ with $\theta \in [1,\infty)$. We then have

$$C(u,v) = \exp(-[(-\log u)^{\theta} + (-\log v)^{\theta}]^{1/\theta}).$$

This is a Gumbel family of copulas.

Example 3. Let $\varphi(t) = -\log((e^{-\theta t} - 1)/(e^{-\theta} - 1))$ with $\theta \in (-\infty,\infty) \setminus \{0\}$. We then have

$$C(u,v) = -\frac{1}{\theta} \log\Big(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\Big).$$

This is a Frank family of copulas.

There are many other Archimedean copulas. For more details, we refer to a book by Nelsen [9].

2.3. Generalized Archimedean copulas

Now in 2007, F. Durante, J.J. Quesada-Molina, and C. Sempi [1] discovered an interesting generalization of Archimedean copulas, which is defined as follows.

Let φ be a convex and strictly decreasing function defined on I and valued in $[0,\infty]$, and let ψ be a continuous and decreasing function with $\psi(1) = 0$ defined on I and valued in $[0,\infty]$. Suppose additionally that $(\psi - \varphi)$ is increasing on I. For such φ, ψ, the function $C_{\varphi,\psi}$ defined on I^2 by

$$C_{\varphi,\psi}(u,v) = \varphi^{[-1]}(\varphi(\min\{u,v\}) + \psi(\max\{u,v\})) \tag{6}$$

is shown to provide a copula.

It is easy to see that if $\varphi \equiv \psi$, then the copula $C_{\varphi,\varphi}$ is reduced to the original Archimedean copula (5) with the generator φ. In this sense, the copula of the form (6) is called a generalized Archimedean copula, which wil be referred to as a GA copula hereafter.

We here mention nontrivial examples of GA copulas, which are presented in [1].

Example 4. Let $\varphi(t) = -\log t$, $\psi(t) = -\log t^{\alpha}$ with $\alpha \in [0,1]$. Then we find that the corresponding copula $C_{\varphi,\psi}$ is

$$C_{\varphi,\psi}(u,v) = \begin{cases} uv^{\alpha} & (u \leq v) \\ u^{\alpha}v & (u \geq v). \end{cases}$$

This family is a member of Cuadras-Augé family of copulas.

Example 5. Let $\varphi(t) = \alpha(1-t)$ $(\alpha \geq 1)$ and let $\psi(t) = 1 - t$. The corresponding copula $C_{\varphi,\psi}$ is

$$\begin{aligned} C_{\varphi,\psi}(u,v) &= \max\Big\{0, \min\{u,v\} - \frac{1}{\alpha}(1 - \max\{u,v\})\Big\} \\ &= \begin{cases} \frac{1}{\alpha}(\alpha\min\{u,v\} + \max\{u,v\} - 1) & (\text{if } \alpha\min\{u,v\} + \max\{u,v\} \geq 1) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

3. Measures of dependence

3.1. Kendall's tau and Spearman's rho

The population version of Kendall's tau and Spearman's rho are known to be the two most common measures of dependence. To quantitatively estimate these dependent relations, several measures of associations have been introduced so far. As widely known examples, we recall the population version of Kendall's tau (τ) and Spearman's rho (ρ).

It is known that τ and ρcan be represented in terms of copulas. Precisely stated, let X and Y be continuous random variables whose copula is C. Then we have

$$\tau_{X,Y} = \tau_C = 4\iint_{I^2} C(u,v)dC(u,v) - 1 = 1 - 4\iint_{I^2} \frac{\partial C}{\partial u}(u,v)\frac{\partial C}{\partial v}(u,v)\,dudv,$$
$$\rho_{X,Y} = \rho_C = 12\iint_{I^2} uvdC(u,v) - 3 = 12\iint_{I^2} C(u,v)\,dudv - 3.$$

If the copula C is Archimedean of the form (5), then τ_C is further computed as follows.

$$\tau_C = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt. \tag{7}$$

3.2. Generalizations

Now we want to extend and generalize the measures $\mathscr{M}$ of dependence involving the copulas, which should be of the form

$$\mathscr{M}_C = \iint_{I^2} f(u,v,C(u,v))dC(u,v), \tag{8}$$

where $f = f(u,v,C)$ is an appropriate smooth positive function, whose detailed assumptions will be specified later. For example, the Kendall's tau is provided by

$$f(u,v,C) = 4C - 1$$

and the Spearman's rho is

$$f(u,v,C) = 12uv - 3.$$

Therefore we understand that the formula (8) is a natural generalization. We here note that $\iint_{I^2} dC(u,v) = 1$.

There should be two conditions imposed on f in (8). One condition is the normalization: There should be

$$|\mathscr{M}_C| \le 1 \qquad \text{for any copula} \quad C. \tag{9}$$

This condition is acomplished by the multiplication of a constant. The other condition is that if $C(u,v) = \Pi(u,v) = uv$, that is, if C is the product copula, then it follows that

$$\mathscr{M}_\Pi = \iint_{I^2} f(u,v,uv)dudv = 0. \tag{10}$$

In other words, the measure of concordance should vanish if random variables are independent.

3.3. Expressions for Archimedean and GA copulas

The main contribution of this article is that we establish a computable formula for (8) with respect to generators as in (7), which makes the calculation much handy.

To proceed further, we restrict ourselves to the case

$$f = f(C) \qquad \text{with} \quad f'(C) \ge 0 \quad \text{for} \quad C \ge 0. \tag{11}$$

We then determine the explicit formula for $\mathscr{M}_C = \iint_{I^2} f(C(u,v))dC(u,v)$ in terms of the generators of Archimedean and GA copulas. The first result reads as follows.

Theorem 1. *Let X and Y be continuous random variables whose copula C_φ is an Archimedean copula with generator φ. Then the measure of dependence (8) with (11) of X and Y is expressed as*

$$\mathscr{M}_C = \int_0^1 f(t)dt + \int_0^1 f'(t)\frac{\varphi(t)}{\varphi'(t)}dt. \tag{12}$$

If $f(C) = 4C-1$, namely, if f gives an ordinary Kendall's tau, then it is easy to see that the formula (12) is reduced to (7).

Sketch of proof. It suffices to consider the case that φ is differentiable. We deduce that

$$\begin{aligned}\mathscr{M}_C &= \int_0^1 f(t)dt - \iint_{I^2} f'(C)\frac{\partial C}{\partial u}\frac{\partial C}{\partial v}dudv \\ &= \int_0^1 f(t)dt - 2\iint_{\{u\le v\}} f'(\varphi^{[-1]}(\varphi(u)+\varphi(v)))\frac{\varphi'(u)\varphi'(v)}{(\varphi'(\varphi^{[-1]}(\varphi(u)+\varphi(v))))^2}dudv.\end{aligned}$$

Performing the change of variables $(u,v) \to (u,t)$, where $\varphi(t) = \varphi(u)+\varphi(v)$ on $\{u \le v\}$, we infer that

$$\begin{aligned}&\iint_{\{u\le v\}} f'(\varphi^{[-1]}(\varphi(u)+\varphi(v)))\frac{\varphi'(u)\varphi'(v)}{(\varphi'(\varphi^{[-1]}(\varphi(u)+\varphi(v))))^2}dudv \\ &= \iint_{\{\varphi^{[-1]}(\varphi(u)+\varphi(v))\le t\le u\}} f'(t)\frac{\varphi'(u)}{\varphi'(t)}dtdu \\ &= \int_0^1 \frac{f'(t)dt}{\varphi'(t)}\Big(\int_t^1 - \int_{t^*}^1\Big)\varphi'(u)du = -\int_0^1 f'(t)\frac{\varphi(t)-\varphi(t^*)}{\varphi'(t)}dt = -\frac{1}{2}\int_0^1 f'(t)\frac{\varphi(t)}{\varphi'(t)}dt,\end{aligned}$$

where t^* is determined by $\varphi(t^*) = 2^{-1}\varphi(t)$. This finishes the proof of Theorem. //

Theorem 2. *Let X and Y be continuous random variables whose copula $C_{\varphi,\psi}$ is a GA copula given by (6). Then the measure of dependence (8) with (11) of X and Y is expressed as*

$$\mathscr{M}_C = \int_0^1 f(t)dt + 2\int_0^1 f'(t)\frac{\varphi(t)-\varphi(t^*)}{\varphi'(t)}dt, \tag{13}$$

where t^* is defined through $\varphi(t^*)+\psi(t^*) = \varphi(t)$.

We remark that if $\varphi \equiv \psi$, then $\varphi(t^*) = 2^{-1}\varphi(t)$ and the formula (12) is recovered from (13).

Sketch of proof. Although the proof is almost the repetition of that of Theorem 1, we exhibit it here for completeness. Assuming that both φ, ψ are differentiable, we derive

$$\begin{aligned}\mathscr{M}_C &= \int_0^1 f(t)dt - \iint_{I^2} f'(C)\frac{\partial C_{\varphi,\psi}}{\partial u}\frac{\partial C_{\varphi,\psi}}{\partial v}dudv \\ &= \int_0^1 f(t)dt - \iint_{\{u\le v\}} f'(\varphi^{[-1]}(\varphi(u)+\psi(v)))\frac{\varphi'(u)\psi'(v)}{(\varphi'(\varphi^{[-1]}(\varphi(u)+\psi(v))))^2}dudv \\ &\quad + \iint_{\{u\ge v\}} f'(\varphi^{[-1]}(\varphi(v)+\psi(u)))\frac{\varphi'(v)\psi'(u)}{(\varphi'(\varphi^{[-1]}(\varphi(v)+\psi(u))))^2}dudv \\ &= \int_0^1 f(t)dt - 2\iint_{\{u\le v\}} f'(\varphi^{[-1]}(\varphi(u)+\psi(v)))\frac{\varphi'(u)\psi'(v)}{(\varphi'(\varphi^{[-1]}(\varphi(u)+\psi(v))))^2}dudv.\end{aligned}$$

Performing the change of variables $(u,v) \to (u,t)$, where $\varphi(t) = \varphi(u) + \psi(v)$ on $\{u \leq v\}$, we infer that

$$\begin{aligned}
&\iint_{\{u\leq v\}} f'(\varphi^{[-1]}(\varphi(u)+\psi(v)))\frac{\varphi'(u)\psi'(v)}{(\varphi'(\varphi^{[-1]}(\varphi(u)+\psi(v))))^2}dudv \\
&= \iint_{\{\varphi^{[-1]}(\varphi(u)+\psi(v))\leq t\leq u\}} f'(t)\frac{\varphi'(u)}{\varphi'(t)}dtdu \\
&= \int_0^1 \frac{f'(t)dt}{\varphi'(t)}\Big(\int_t^1 - \int_{t^*}^1\Big)\varphi'(u)du = -\int_0^1 f'(t)\frac{\varphi(t)-\varphi(t^*)}{\varphi'(t)}dt,
\end{aligned}$$

where t^* is determined by $\varphi(t^*) + \psi(t^*) = \varphi(t)$. This completes the proof of Theorem. //

4. Examples

In this section, we compute the generalized measures of dependence for the family of Archimedean and GA copulas. As an example, we treat the case

$$f_n(C) = \frac{(n+1)^2}{n}C^n - \frac{1}{n} \qquad (n = 1,2,\cdots), \tag{14}$$

and show that by before listed Examples how our formulas (12)(13) works. It is to be noted that $f_n(C)$ above (14) satisfies the conditions (9)(10)(11) in light of the Fréchet-Hoeffding bounds (4).

Example 1 (continued). Since $\varphi(t) = \theta^{-1}(t^{-\theta} - 1)$ ($\theta \in [-1,\infty)\setminus\{0\}$), we derive

$$\mathscr{M} = 1 - \int_0^1 \frac{(n+1)^2}{\theta}(t^n - t^{n+\theta})dt = 1 - \frac{n+1}{n+\theta+1}.$$

Example 2 (continued). Since $\varphi(t) = (\log t)^\theta$ ($\theta \in [1,\infty)$), we infer that

$$\mathscr{M} = 1 - \int_0^1 \frac{(n+1)^2}{\theta}t^n \log t dt = 1 - \frac{1}{\theta},$$

which is independent of n.

Example 3 (continued). Since $\varphi(t) = -\log((e^{-\theta t}-1)/(e^{-\theta}-1))$ ($\theta \in (-\infty,\infty)\setminus\{0\}$), we deduce that

$$\mathscr{M} = 1 - \int_0^1 \frac{(n+1)^2}{\theta}t^{n-1}(1-e^{\theta t})\log\Big(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\Big)dt.$$

There seems to be no way to advance this integral further.

Example 4 (continued). Since $\varphi(t) = -\log t$, $\psi(t) = -\log t^\alpha$, we see that $t^* = t^{1/(\alpha+1)}$. Consequently we have

$$\mathscr{M} = 1 + 2\int_0^1 (n+1)^2 t^n \log t^{(1-1/(\alpha+1))}dt = 1 - \frac{2\alpha}{\alpha+1},$$

which is independent of n.

Example 5 (continued). Since $\varphi(t) = \alpha(1-t)$, $\psi(t) = 1-t$, we find that $1-t^* = \alpha(1-t)/(\alpha+1)$. We thus obtain

$$\mathscr{M} = 1 - 2\int_0^1 (n+1)^2 t^{n-1}\frac{1-t}{\alpha+1}dt = 1 - \frac{2(n+1)}{n(\alpha+1)}.$$

We note that the same results are also obtained by direct calculation.

5. Conclusion

We have introduced and studied a generalized measure of dependence via the use of copula functions. Since copulas give flexible tools for understanding possibly nonlinear relations between risk factors, our generalizations may be useful. We have derived the formula of this generalized measure of association for the family of Archimedean and generalized Archimedean copulas. The formula brings us to an alternative way of computation of the measure of concordance. Examples show that some copulas are certainly under the effect of current generalization. We hope that our achievement will enrich the application of copulas.

The theory of copulas has been widely developed from various aspects so far. One of the authors recently introduce the notion of evolution of copulas [5][6][12], which is relatively a new concept of time related copulas. We remark that the theory of dynamic copula [10] is already employed in the literature, which is different from our evolution of copulas. In any case, it would be interesting to investigate the time dependent problem of the measures of association; the dependence structure is naturally supposed to be varied with time variable. We will return to deal with this issue in the near future.

Acknowledgements

The second author (N. I.) is supported in part by Grant-in-Aid for Scientific Research (Kakenhi) No. 24243031 from Japan Society for the Promotion of Science (JSPS).

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